COE CST Eleventh Annual Technical Meeting

399-UCF
Efficient Uncertainty Quantification, Probability of Collision and Benchmarking

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Agenda

- Team Members
- Task Description
- Schedule
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- Results
- Conclusions and Future Work

Team Members

- People

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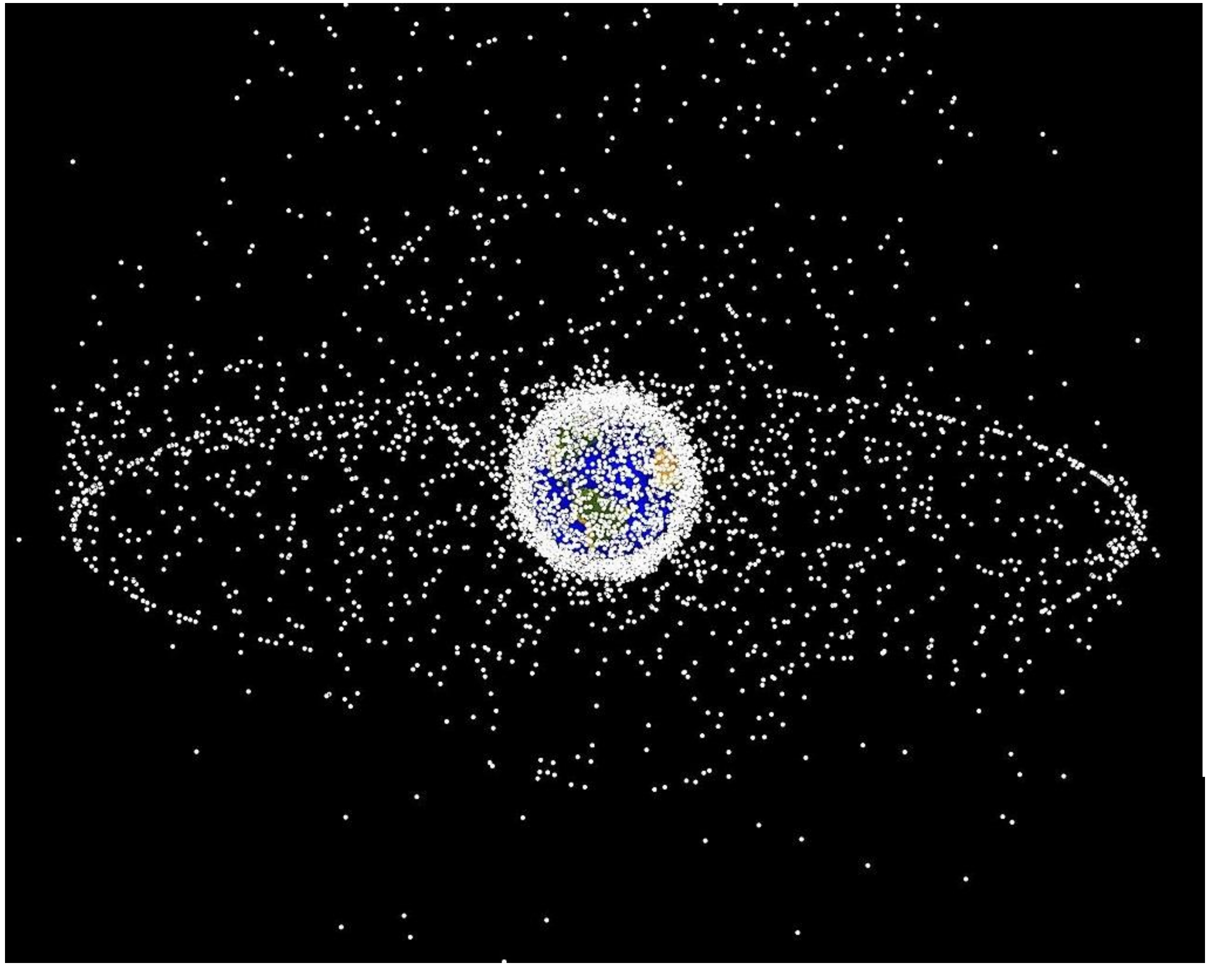
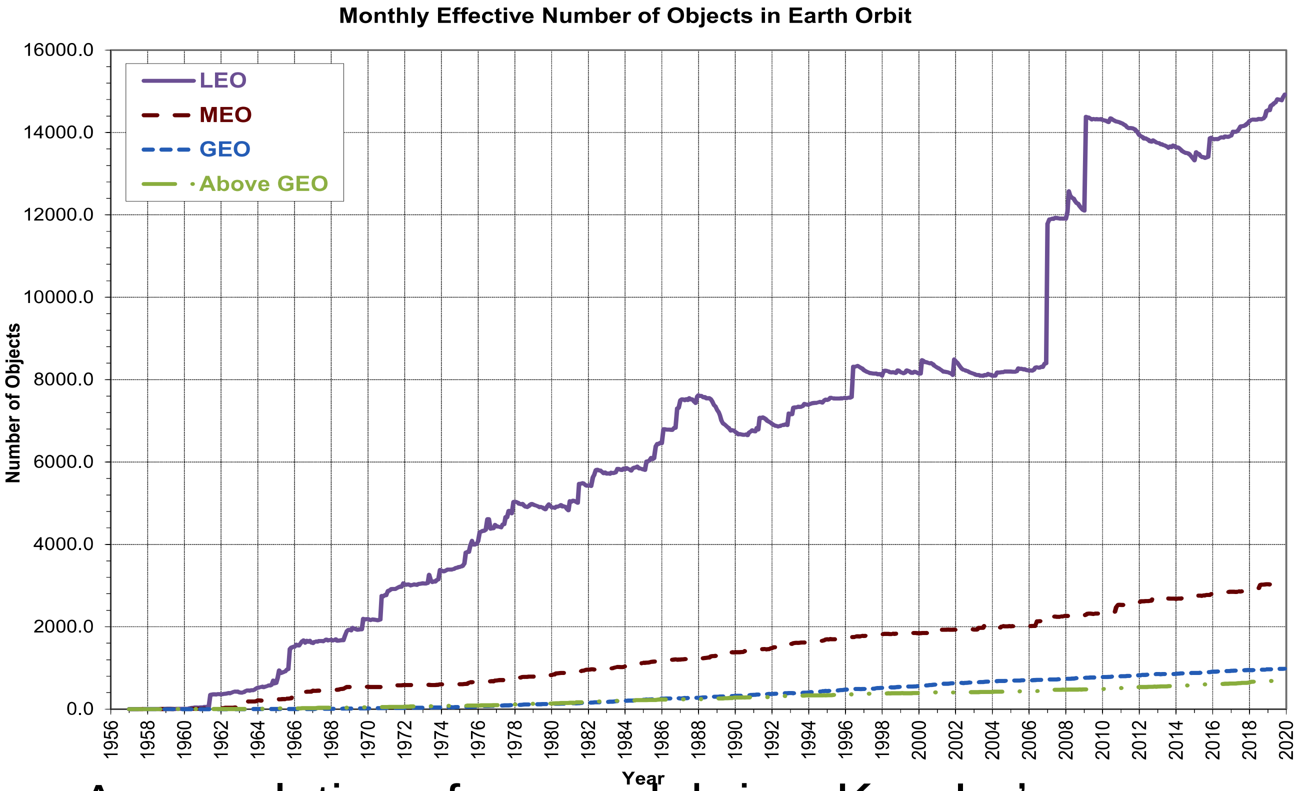
Task Description





- Accumulation of space debris – Kessler's Syndrome – Sustainability of the space environment
- More and more constellations in Earth orbit – SpaceX, OneWeb, India, China, etc.

Our Approaches for UQ

- **Probability Density Function (PDF) via Higher Order State Transition Tensors**
  - Evolution of uncertainties

$$\delta x = \phi_1 \delta x_0 + \phi_2 \delta x_0 \delta x_0 + \cdots$$

  - Knowing the probability distribution of δx_0, the posterior PDF is given by,

$$P_{\delta x}(\delta x) = P_{\delta x_0}(\delta x_0) \left| \det\left(\frac{\partial g^{-1}(\delta x)}{\partial(\delta x)} \right) \right|$$

  - Where, $g^{-1}(\delta x)$ is the Taylor series reversion.

$$\delta \boldsymbol{x}_0 = \Psi_1 \delta \boldsymbol{x} + \frac{1}{2} \Psi_2 \delta \boldsymbol{x} \delta \boldsymbol{x} + \frac{1}{3!} \Psi_3 \delta \boldsymbol{x} \delta \boldsymbol{x} \delta \boldsymbol{x} + \ldots$$

$$\delta \boldsymbol{x}_0' = \Psi_1 + \frac{1}{2} \Psi_2 \delta \boldsymbol{x} + \frac{1}{3!} \Psi_3 \delta \boldsymbol{x} \delta \boldsymbol{x} + \ldots$$

$$P_{\delta \boldsymbol{x}}(\delta \boldsymbol{x}) = P_{\delta \boldsymbol{x}_0}(\delta \boldsymbol{x}_0) \left| \det\left(\delta \boldsymbol{x}_0'\right) \right|$$

Analytic Continuation Technique

- Analytic Continuation is an integration method applied to solve fundamental problems in Astrodynamics.
- This method has been proven to be highly precise and computationally efficient in orbit propagation.
- The full spherical harmonics gravity model and atmospheric drag model were also incorporated with Analytic Continuation method.

$$f = \boldsymbol{r}.\boldsymbol{r} \text{ and } g_p = f^{-\frac{p}{2}}$$

$$\boldsymbol{r}_0^{(2)} = -\mu \frac{\boldsymbol{r}_0}{(\boldsymbol{r}_0.\boldsymbol{r}_0)^{3/2}} = -\mu\, \boldsymbol{r}_0 f^{-\frac{3}{2}} = -\mu\, \boldsymbol{r}_0 g_3$$

Analytic Continuation - State Variables

- Taylor series expansion to obtain position and velocity:

$$\boldsymbol{r}(t_0 + dT) = \boldsymbol{r}_0 + \sum_{m=1}^{n} \boldsymbol{r}_0^{(m)} \frac{dT^{(m)}}{m!}$$

$$\boldsymbol{r}^{(1)}(t_0 + dT) = \boldsymbol{r}_0^{(1)} + \sum_{m=2}^{n} \boldsymbol{r}_0^{(m)} \frac{dT^{(m-1)}}{(m-1)!}$$

- The recursive equations to calculate $\boldsymbol{r}_0^{(n)}, f^{(n)}\ and\ g_p^{(n)}$:

$$\boldsymbol{r}_0^{(n+2)} = -\mu \sum_{m=0}^{n} \binom{n}{m} \boldsymbol{r}_0^{(m)} g_3^{(n-m)} \text{ and } f^{(n)} = \sum_{m=0}^{n} \binom{n}{m} \boldsymbol{r}_0^{(m)} . \boldsymbol{r}_0^{(n-m)}$$

$$g_p^{(n+1)} = -\frac{1}{f} \left\{ \frac{p}{2} f^{(1)} g_p^{(n)} + \sum_{m=1}^{n} \binom{n}{m} \left(\frac{p}{2} f^{(m+1)} g_p^{(n-m)} + f^{(m)} g_p^{(n-m+1)} \right) \right\}$$

Analytic Continuation – State Transition Tensors

- Index based First and Second order State Transition Tensors:

$$\phi_{ij}^{1} = \frac{\partial \chi_i}{\partial \chi_{0j}} \text{ and } \phi_{ijk}^{2} = \frac{\partial^2 \chi_i}{\partial \chi_{0j} \partial \chi_{0k}}$$

where, χ_i is the i-th element of the state vector, $\chi = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$.

- Taylor series expansion of the terms of the State Transition Tensors:

$$\phi_{i=1,\dots,3,jk}^{2}(t + dT, t) = \frac{\partial^2 \chi_i(t+dT)}{\partial \chi_j(t) \partial \chi_k(t)} = \frac{\partial^2 \chi_i(t)}{\partial \chi_j(t) \partial \chi_k(t)} + \sum_{m=1}^{n} \frac{\partial^2 \chi_i^{(m)}(t)}{\partial \chi_j(t) \partial \chi_k(t)} \frac{dT^{(m)}}{(m)!}$$

$$\phi_{i=4,\dots,6,jk}^{2}(t + dT, t) \frac{\partial^2 \chi_i(t+dT)}{\partial \chi_j(t) \partial \chi_k(t)} = \frac{\partial^2 \chi_i(t)}{\partial \chi_j(t) \partial \chi_k(t)} + \sum_{m=2}^{n} \frac{\partial^2 \chi_i^{(m)}(t)}{\partial \chi_j(t) \partial \chi_k(t)} \frac{dT^{(m-1)}}{(m-1)!}$$

Schedule

Task	Time Frame
Develop Analytic Continuation for arbitrary order perturbed state transition tensors for accurate error propagation	Fall 2020
Develop estimation framework for space-based surveillance and tracking utilizing the perturbed STM/STT.	Fall 2021
Computing Probability of collisions of RSOs via two approaches + Benchmarking problems	Spring/Summer 2022

Goals

- Accurate and efficient approaches to quantify uncertainty and compute probability of collision for RSOs
- Benchmarking platform for other methods to provide synthetic or real cases and compare results
- Sustainability of the space environment
- Tools to predict space debris trajectories and potential hazardous events to various operators
- Accurate orbit prediction for newly deployed constellations and their potential collisions with debris and/or other RSOs.

Results

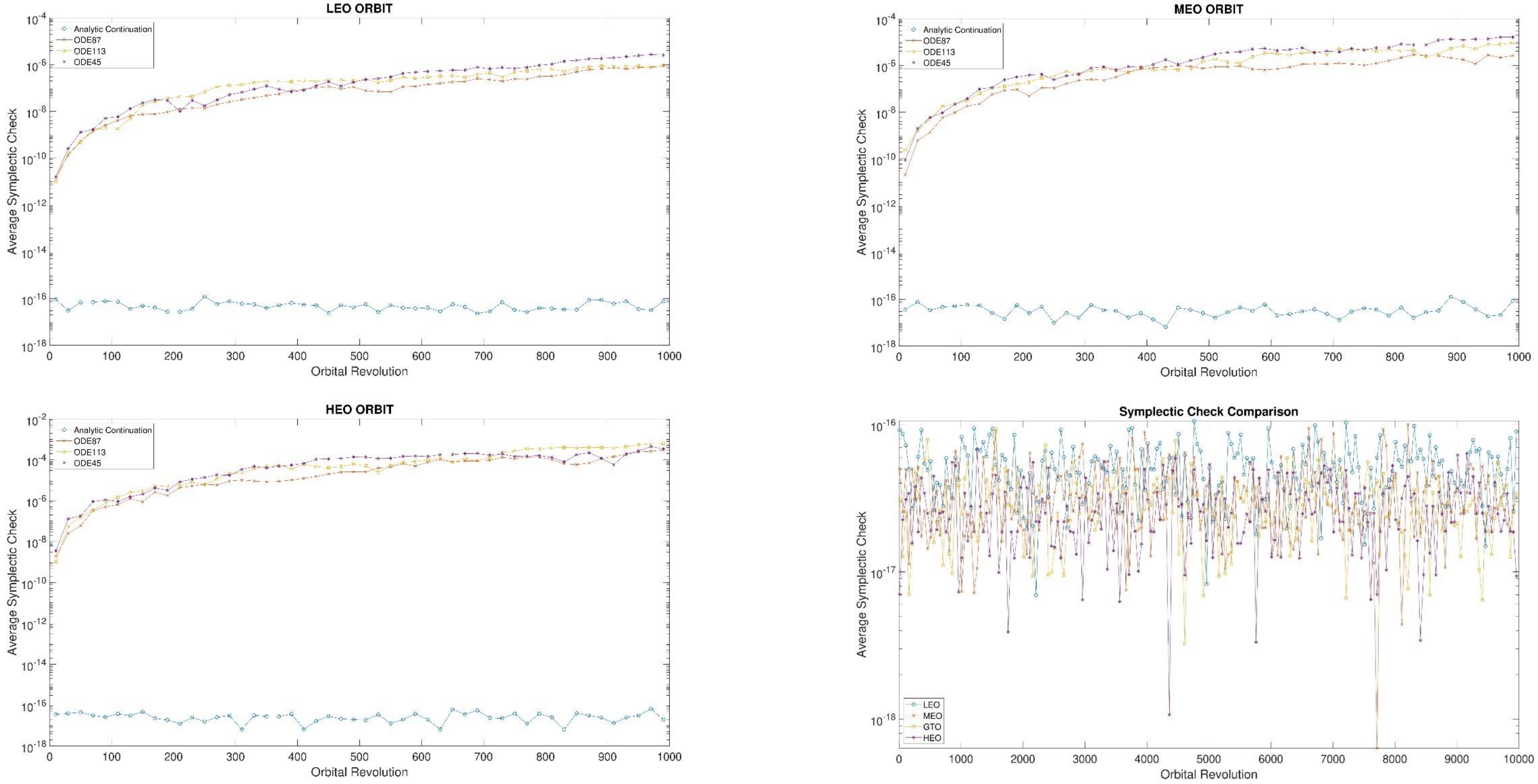


Fig: Symplectic Error in $J_2 – J_6$ gravity perturbed orbits and comparison with MATLAB ODE suite

Results (continued)

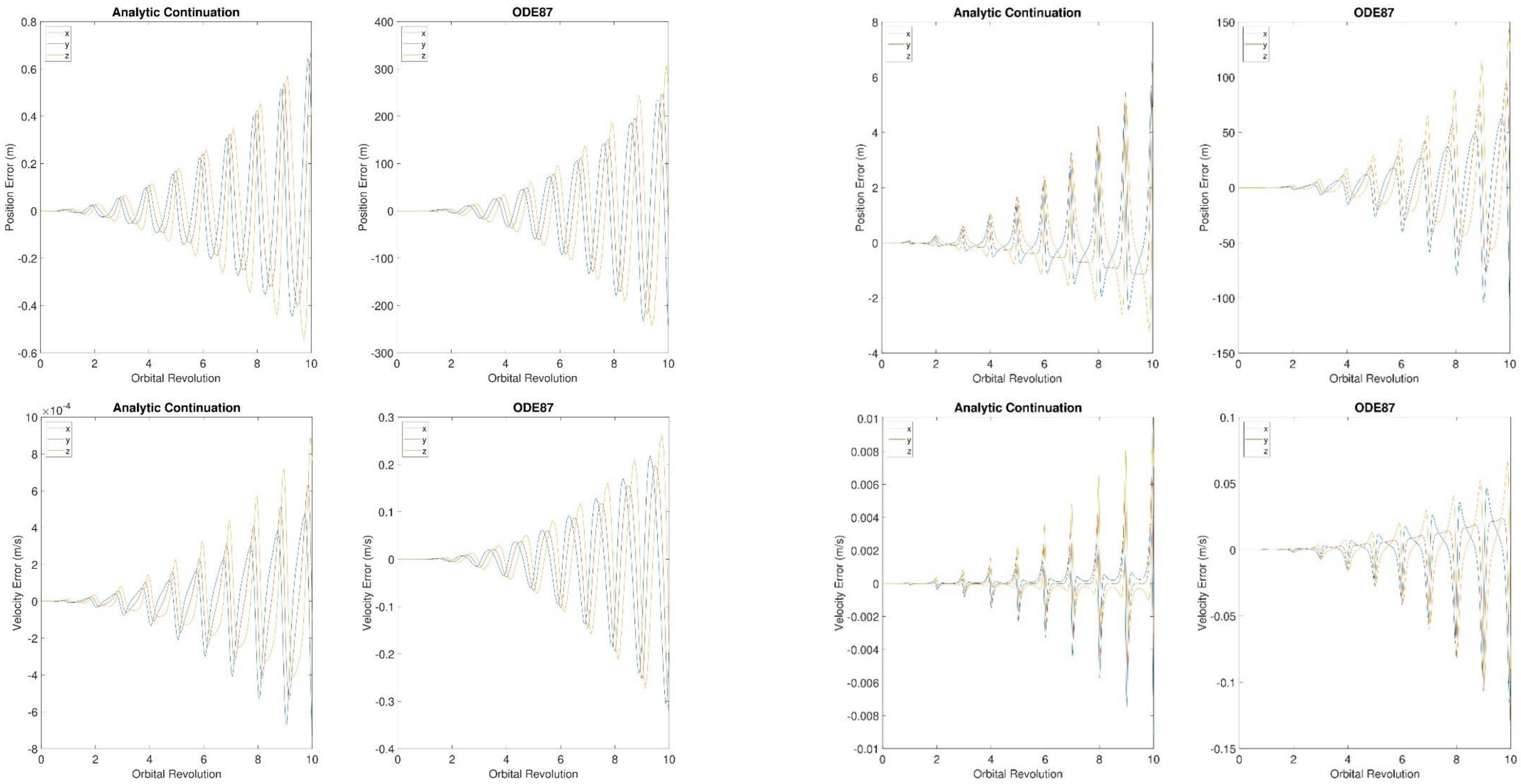


Fig: Linear prediction error of states of $J_2 – J_6$ gravity and drag perturbed LEO and MEO orbit using Analytic Continuation and comparison with ODE87

Results (continued)

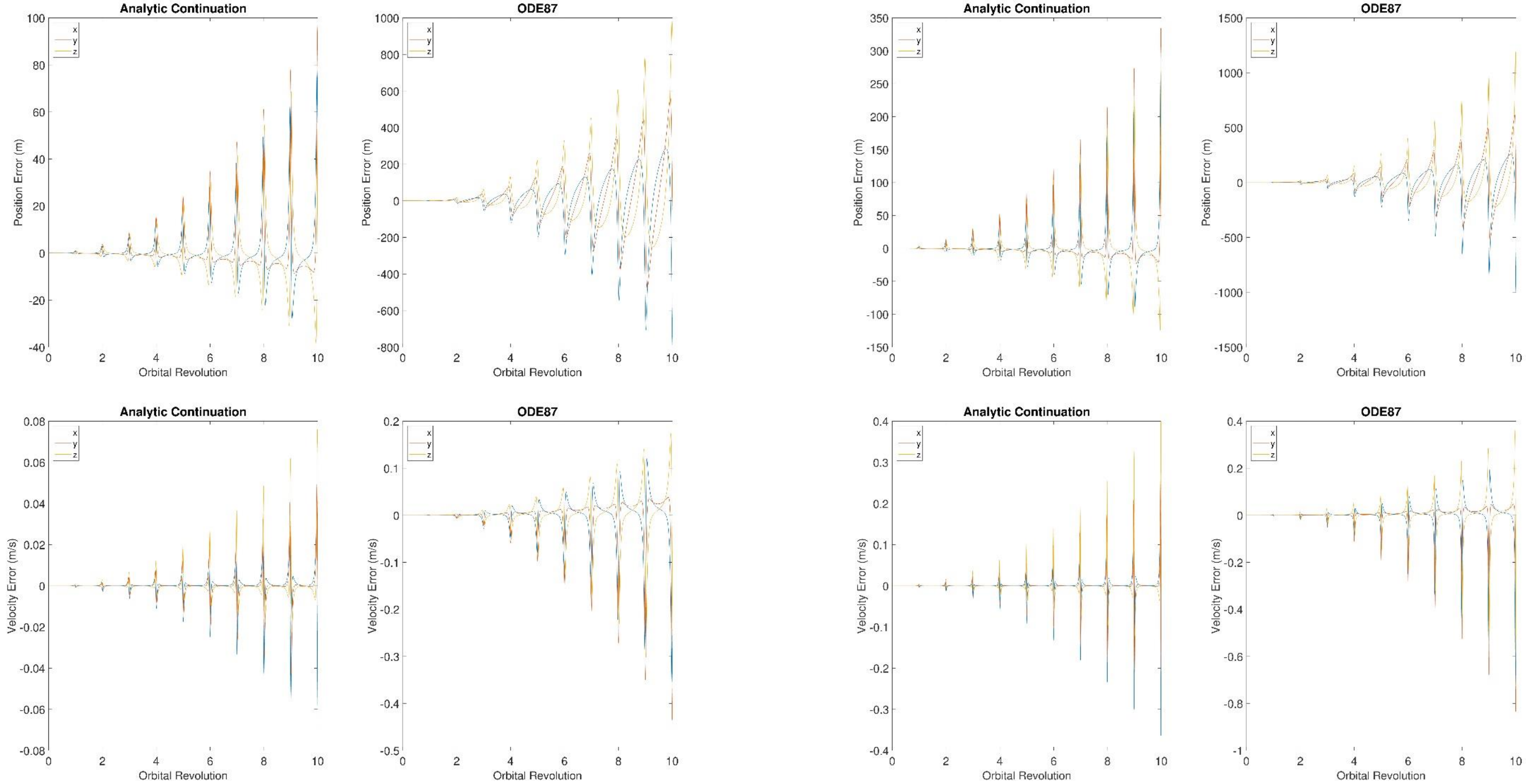


Fig: Linear prediction error of states of $J_2 - J_6$ gravity and drag perturbed GTO and HEO orbit using Analytic Continuation and comparison with ODE87

Results (continued)

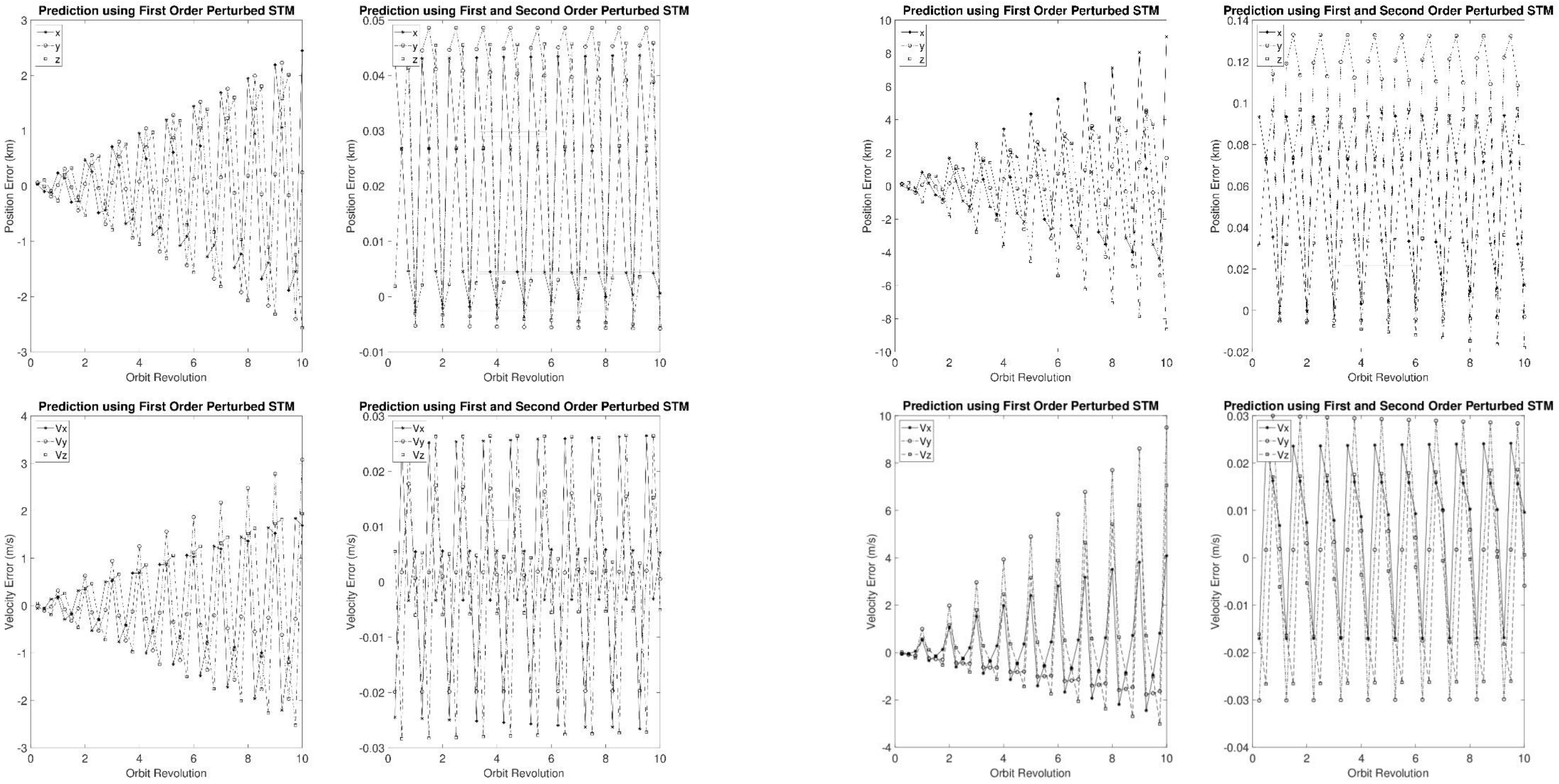


Fig: 2nd order prediction error improvement of states of J_2 perturbed LEO and MEO orbit using Second Order State Transition Tensor derived using Analytic Continuation technique

Results (continued)

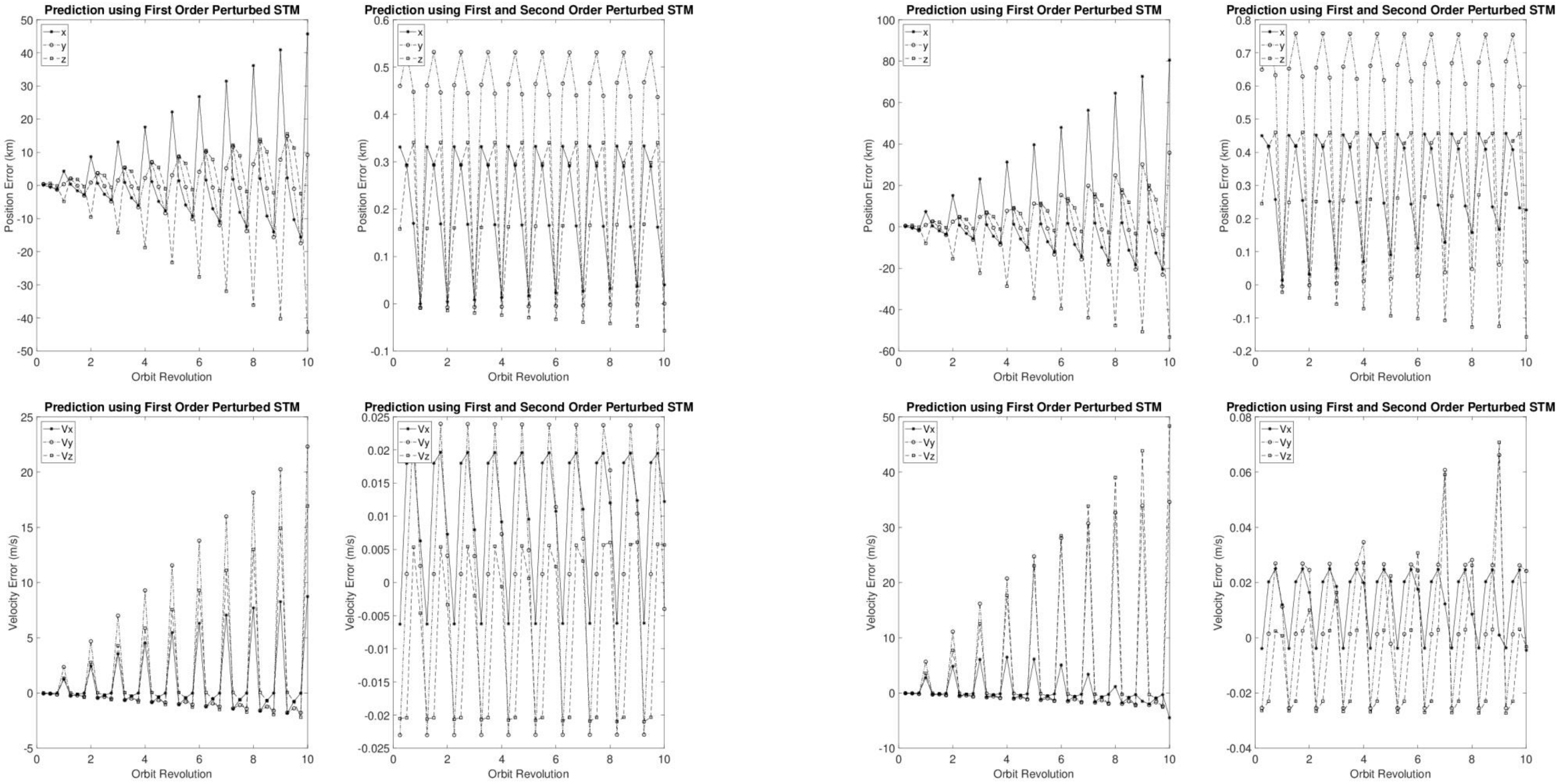
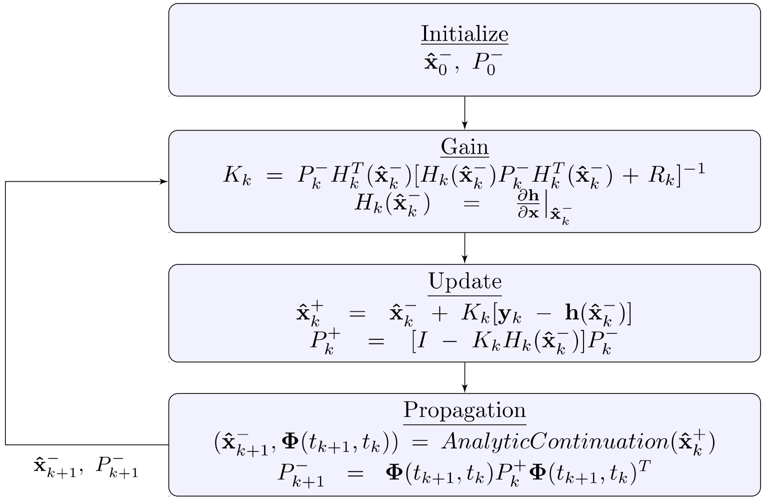


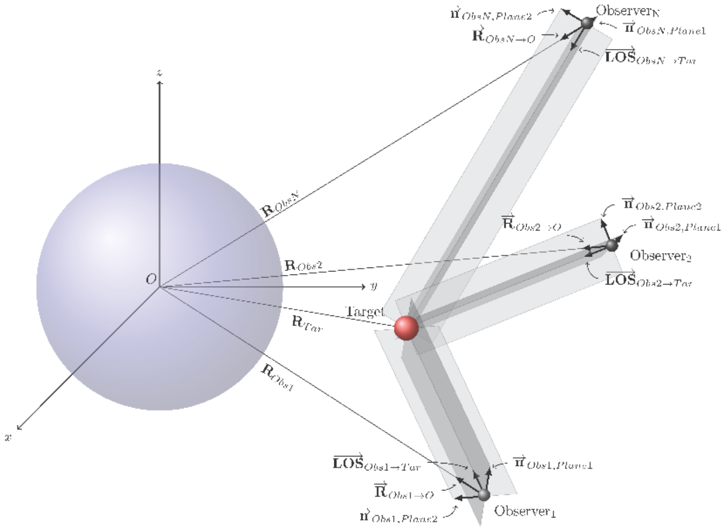
Fig: 2nd order prediction error improvement of states of J_2 perturbed GTO and HEO orbit using Second Order State Transition Tensor derived using Analytic Continuation technique

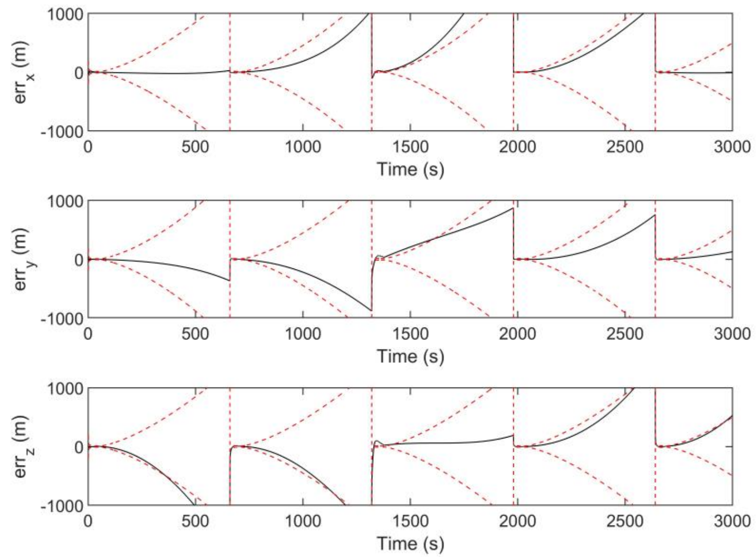
Application – Accurate Orbit Estimation with Sparse Measurements for Space-Based Surveillance & Tracking (SBSST)

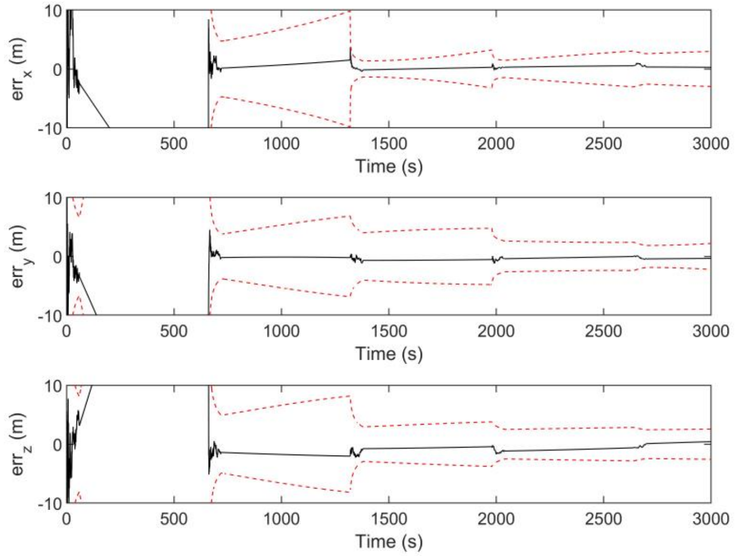
- Analytic Continuation Extended Kalman Filter (AC-EKF)



$$H(\hat{\mathbf{X}}_{Tar_k}) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\bigg|_{\hat{\mathbf{X}}_{Tar_k}} = \begin{bmatrix} \frac{1}{(z_{Obs1_k} - \hat{z}_{Tar_k})} & 0 & \frac{(\hat{x}_{Tar_k} - x_{Obs1_k})}{(z_{Obs1_k} - \hat{z}_{Tar_k})^2} & 0 & 0 & 0 \\ 0 & \frac{1}{(z_{Obs1_k} - \hat{z}_{Tar_k})} & \frac{(\hat{y}_{Tar_k} - y_{Obs1_k})}{(z_{Obs1_k} - \hat{z}_{Tar_k})^2} & 0 & 0 & 0 \\ & \vdots & & & & \\ \frac{1}{(z_{ObsN_k} - \hat{z}_{Tar_k})} & 0 & \frac{(\hat{x}_{Tar_k} - x_{ObsN_k})}{(z_{ObsN_k} - \hat{z}_{Tar_k})^2} & 0 & 0 & 0 \\ 0 & \frac{1}{(z_{ObsN_k} - \hat{z}_{Tar_k})} & \frac{(\hat{y}_{Tar_k} - y_{ObsN_k})}{(z_{ObsN_k} - \hat{z}_{Tar_k})^2} & 0 & 0 & 0 \end{bmatrix}$$







AC-EKF Accuracy & Efficiency Comparison

A Small Formation Performing SBSST

	Pos. RMSE (m)	Vel. RMSE (m/s)	Condition Number
	Keplerian Two-Body Motion		
Two-body Assumption w/o P.N.	1.785	0.888	5.920×10^{11}
AC-EKF	1.792	0.888	5.931×10^{11}
	Two-Body Motion Gravity Perturbed		
Two-body Assumption w/o P.N.		*Diverges*	
Two-body Assumption w/ P.N.	5.680	1.672	6.395×10^{11}
AC-EKF	1.646	1.348	6.592×10^{11}
	Two-Body Motion Gravity and Drag Perturbed		
Two-body Assumption w/o P.N.		*Diverges*	
Two-body Assumption w/ P.N.	5.644	0.926	5.480×10^{11}
AC-EKF	1.542	0.916	5.602×10^{11}
	Two-Body Higher Order Gravity and Drag Perturbed		
Two-body Assumption w/o P.N.		*Diverges*	
Two-body Assumption w/ P.N.	9.470	0.547	6.896×10^{11}
Analytic Continuation w/o P.N.		*Diverges*	
Analytic Continuation w/ P.N.	4.592	0.518	6.916×10^{11}

Computational Efficiency

	Relative Time	Num. Steps	Pos. RMSE (m)	Vel. RMSE (m/s)
	$\Delta t = 0.5$ s			
F&G-EKF	0.402	1	5.995	1.042
ODE45-EKF w/ P.N.	7.679	12	5.953	0.967
RK4-EKF	1.000	2	1.542	0.916
ODE45D-EKF	11.102	10	1.542	0.916
ODE45-EKF	11.409	10	1.542	0.916
AC-EKF	2.822	1	1.542	0.916
	$\Delta t = 200.0$ s			
F&G-EKF	0.073	1	928.977	4.114
ODE45-EKF w/ P.N.	2.692	100	922.227	4.109
RK4-EKF	1.000	40	25.827	0.106
ODE45D-EKF	1.703	13	25.827	0.106
ODE45-EKF	3.018	39	25.827	0.106
AC-EKF	0.158	1	25.827	0.106

Publications

- Tasif, T.H., Elgohary, T.A.: A high order analytic continuation technique for the perturbed two-body problem state transition matrix, Advances in Astronautical Sciences: **AAS/AIAA Space Flight Mechanics Meeting** (2019)
- Tasif, T.H., Elgohary, T.A.: An adaptive analytic continuation technique for the computation of the higher order state transition tensors for the perturbed two-body problem, **AIAA Scitech 2020** Forum, p. 0958 (2020)
- Tasif, T.H., Elgohary, T.A.: An adaptive analytic continuation method for computing the perturbed two-body problem state transition matrix, **The Journal of the Astronautical Sciences** (2020).
- Tasif, Tahasinul H.; Hippelheuser, James; and Elgohary, Tarek A., "Analytic Continuation Extended Kalman Filter Framework for Space-Based Inertial Orbit Estimation via a Network of Observers", **IAA Space Traffic Management Conference**, January 26 – 27, 2021.
- Tasif, Tahsinul Haque; and Elgohary, Tarek A., "A Computation Process for the Higher Order State Transition Tensors of the Gravity and Drag Perturbed Two-Body Problem using Adaptive Analytic Continuation Technique", **The International Conference on Computational and Experimental Engineering and Sciences (ICCES 2022)**, January 2022.
- Tasif, Tahsinul H.; Hippelheuser, James; and Elgohary, Tarek A., "An Analytic Continuation Extended Kalman Filter Framework for Perturbed Orbit Estimation Using a Network of Space-Based Observers with Angles-Only Measurements", **Astrodynamics** (2022). **In Press.**

Conclusions and Future Work

- Implementation of Spherical Harmonics Gravity and drag perturbations for State Transition Matrix and Higher Order State Transition Tensors.
- A robust estimation framework for multi-observer space-based surveillance and tracking in the absence of continuous measurements.
- The results of the current research work will be extended to solve uncertainty quantification of states over time and perturbed Multi Revolution Lambert’s Problem.