COE CST Tenth Annual Technical Meeting

399-UCF

Efficient Uncertainty Quantification, Probability of Collision and Benchmarking

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Agenda

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- Results
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Team Members

- People

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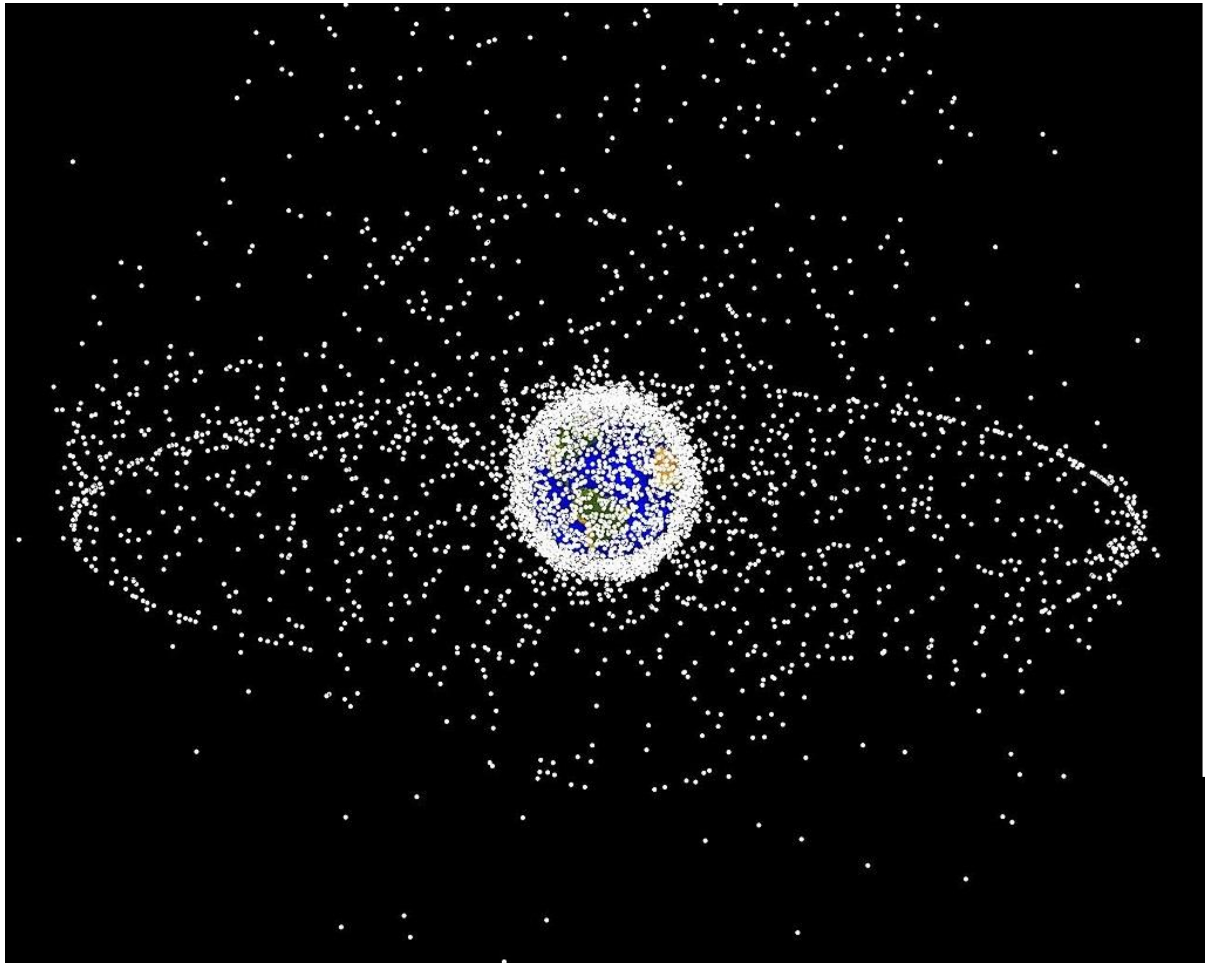
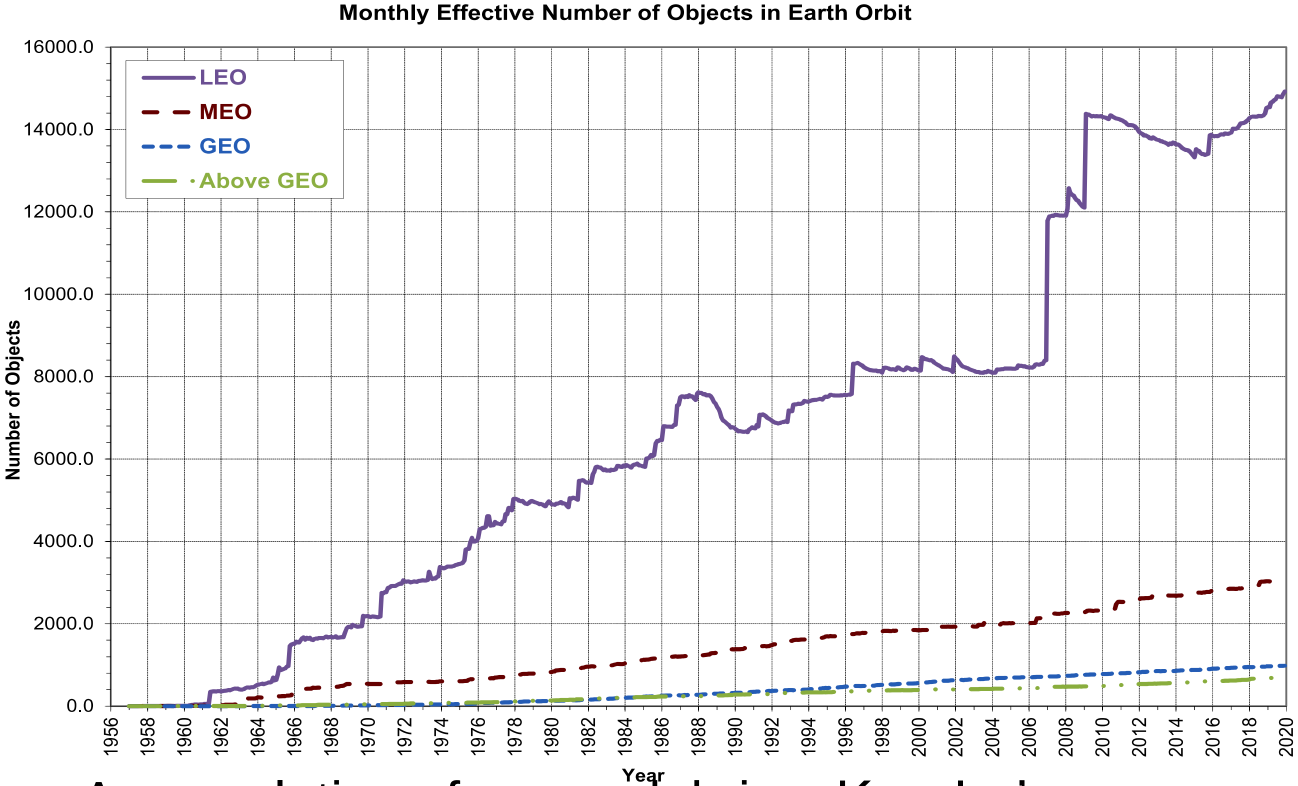
Task Description





- Accumulation of space debris – Kessler's Syndrome – Sustainability of the space environment
- More and more constellations in Earth orbit – SpaceX, OneWeb, India, China, etc.

Two Approaches for UQ

- **Probability Density Function (PDF) via Higher Order State Transition Tensors**
  - Evolution of uncertainties

$$\delta x = \phi_1 \delta x_0 + \phi_2 \delta x_0 \delta x_0 + \cdots$$

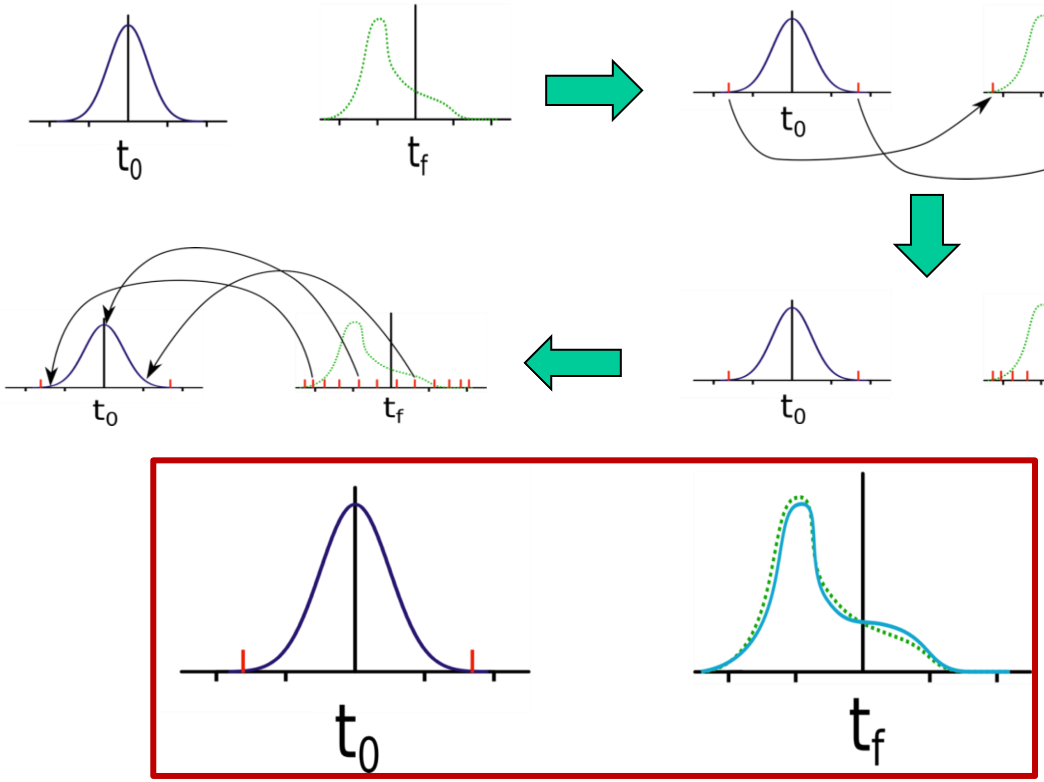
  - Knowing the probability distribution of δx_0, the posterior PDF is given by,

$$P_{\delta x}(\delta x) = P_{\delta x_0}(\delta x_0)\left|\det\left(\frac{\partial g^{-1}(\delta x)}{\partial(\delta x)}\right)\right|$$

  - Where, $g^{-1}(\delta x)$ is the Taylor series reversion.

Focus of Today's Talk

- **Probability Density Function (PDF) Approximation Techniques**



Analytic Continuation Technique

- Analytic Continuation is an integration method applied to solve fundamental problems in Astrodynamics.
- This method has been proven to be highly precise and computationally efficient in orbit propagation.
- The full spherical harmonics gravity model and atmospheric drag model were also incorporated with Analytic Continuation method.

$$f = \boldsymbol{r}.\boldsymbol{r} \text{ and } g_p = f^{-\frac{p}{2}}$$

$$\boldsymbol{r}_0^{(2)} = -\mu \frac{\boldsymbol{r}_0}{(\boldsymbol{r}_0.\boldsymbol{r}_0)^{3/2}} = -\mu\, \boldsymbol{r}_0 f^{-\frac{3}{2}} = -\mu\, \boldsymbol{r}_0 g_3$$

Analytic Continuation - State Variables

- Taylor series expansion to obtain position and velocity:

$$\boldsymbol{r}(t_0 + dT) = \boldsymbol{r}_0 + \sum_{m=1}^{n} \boldsymbol{r}_0^{(m)} \frac{dT^{(m)}}{m!}$$

$$\boldsymbol{r}^{(1)}(t_0 + dT) = \boldsymbol{r}_0^{(1)} + \sum_{m=2}^{n} \boldsymbol{r}_0^{(m)} \frac{dT^{(m-1)}}{(m-1)!}$$

- The recursive equations to calculate $\boldsymbol{r}_0^{(n)}, f^{(n)}\ and\ g_p^{(n)}$:

$$\boldsymbol{r}_0^{(n+2)} = -\mu \sum_{m=0}^{n} \binom{n}{m} \boldsymbol{r}_0^{(m)} g_3^{(n-m)} \text{ and } f^{(n)} = \sum_{m=0}^{n} \binom{n}{m} \boldsymbol{r}_0^{(m)} . \boldsymbol{r}_0^{(n-m)}$$

$$g_p^{(n+1)} = -\frac{1}{f} \left\{ \frac{p}{2} f^{(1)} g_p^{(n)} + \sum_{m=1}^{n} \binom{n}{m} \left(\frac{p}{2} f^{(m+1)} g_p^{(n-m)} + f^{(m)} g_p^{(n-m+1)} \right) \right\}$$

Analytic Continuation – State Transition Tensors

- Index based First and Second order State Transition Tensors:

$$\phi^1_{ij} = \frac{\partial \chi_i}{\partial \chi_{0j}} \text{ and } \phi^2_{ijk} = \frac{\partial^2 \chi_i}{\partial \chi_{0j} \partial \chi_{0k}}$$

where, χ_i is the i-th element of the state vector, $\chi = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$.

- Taylor series expansion of the terms of the State Transition Tensors:

$$\phi^2_{i=1,\ldots,3,jk}(t + dT, t) = \frac{\partial^2 \chi_i(t+dT)}{\partial \chi_j(t) \partial \chi_k(t)} = \frac{\partial^2 \chi_i(t)}{\partial \chi_j(t) \partial \chi_k(t)} + \sum_{m=1}^{n} \frac{\partial^2 \chi_i^{(m)}(t)}{\partial \chi_j(t) \partial \chi_k(t)} \frac{dT^{(m)}}{(m)!}$$

$$\phi^2_{i=4,\ldots,6,jk}(t + dT, t) \frac{\partial^2 \chi_i(t+dT)}{\partial \chi_j(t) \partial \chi_k(t)} = \frac{\partial^2 \chi_i(t)}{\partial \chi_j(t) \partial \chi_k(t)} + \sum_{m=2}^{n} \frac{\partial^2 \chi_i^{(m)}(t)}{\partial \chi_j(t) \partial \chi_k(t)} \frac{dT^{(m-1)}}{(m-1)!}$$

Schedule

Task	Time Frame
Develop Analytic Continuation for arbitrary order perturbed state transition tensors for accurate error propagation	Fall 2020
Posterior PDF approximation via high-order state transition tensors and computation of probability of collisions	Spring/Summer 2021
Orthogonal Probability Approximation for posterior PDF with parametric uncertainty	Summer/Fall 2021
Computing Probability of collisions of RSOs via two approaches + Benchmarking problems	Spring/Summer 2022

Goals

- Accurate and efficient approaches to quantify uncertainty and compute probability of collision for RSOs
- Benchmarking platform for other methods to provide synthetic or real cases and compare results
- Sustainability of the space environment
- Tools to predict space debris trajectories and potential hazardous events to various operators
- Accurate orbit prediction for newly deployed constellations and their potential collisions with debris and/or other RSOs.

Results

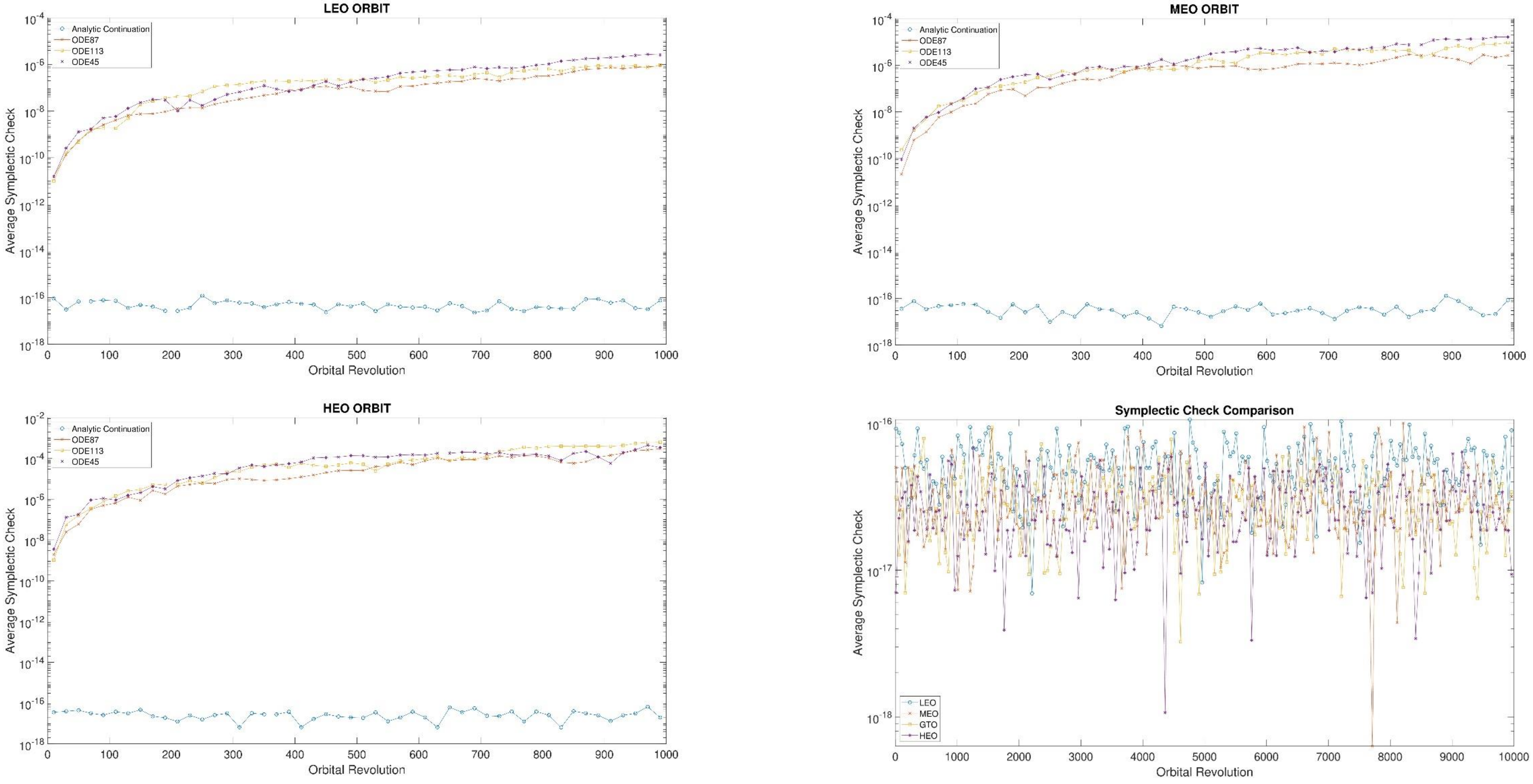


Fig: Symplectic Error in $J_2 - J_6$ gravity perturbed orbits and comparison with MATLAB ODE suite

Results (continued)

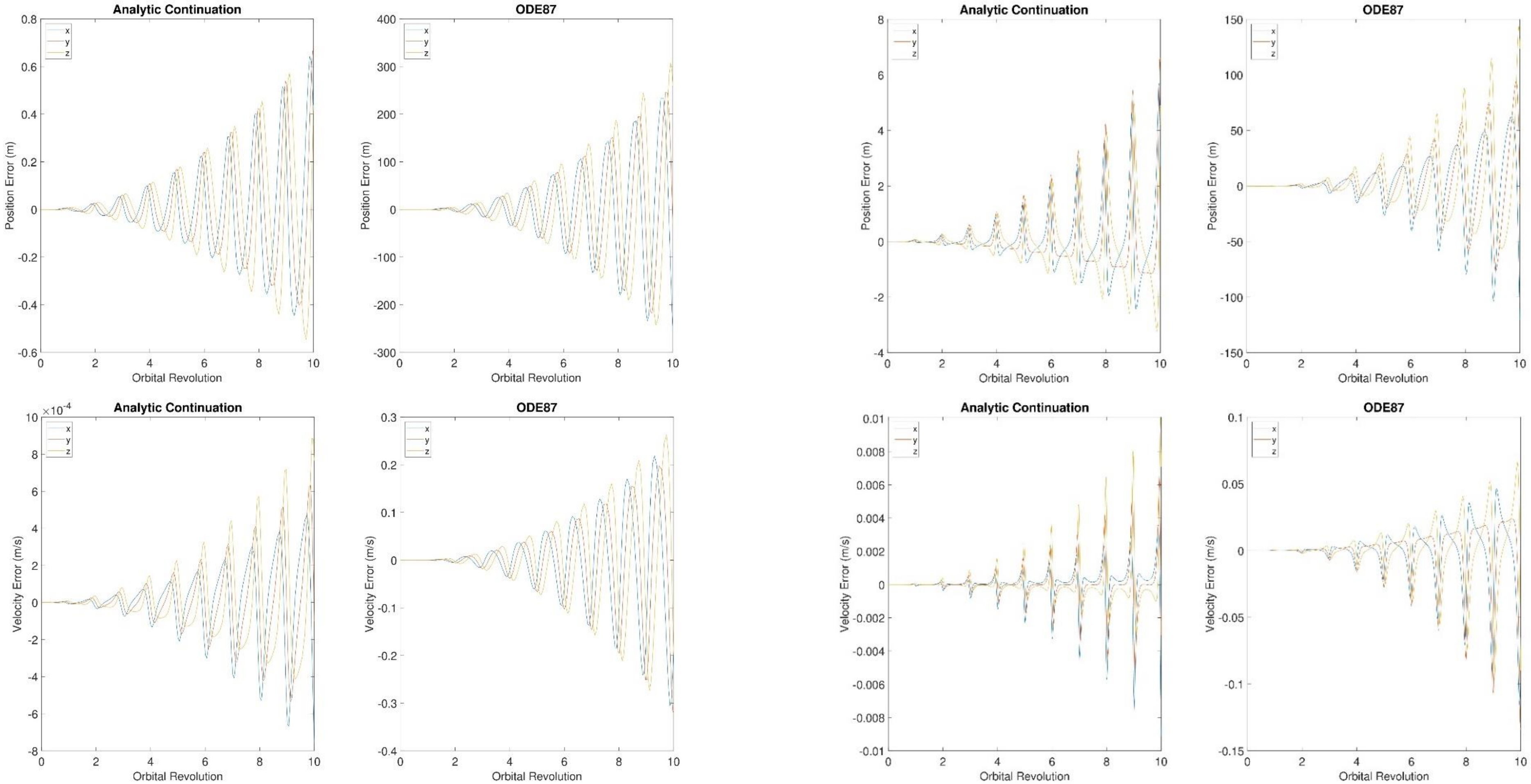


Fig: Linear prediction error of states of $J_2 - J_6$ gravity and drag perturbed LEO and MEO orbit using Analytic Continuation and comparison with ODE87

Results (continued)

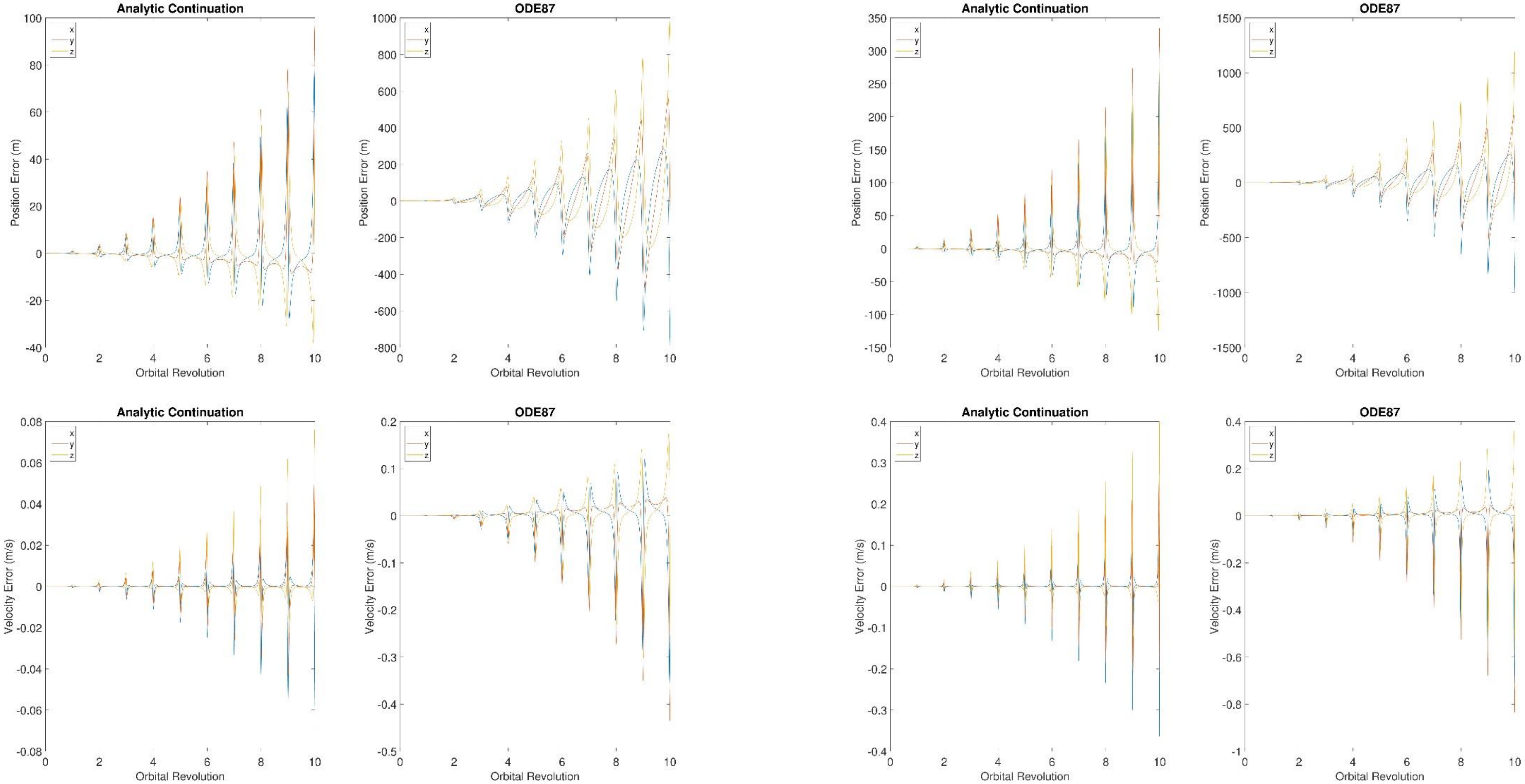


Fig: Linear prediction error of states of $J_2 - J_6$ gravity and drag perturbed GTO and HEO orbit using Analytic Continuation and comparison with ODE87

Results (continued)

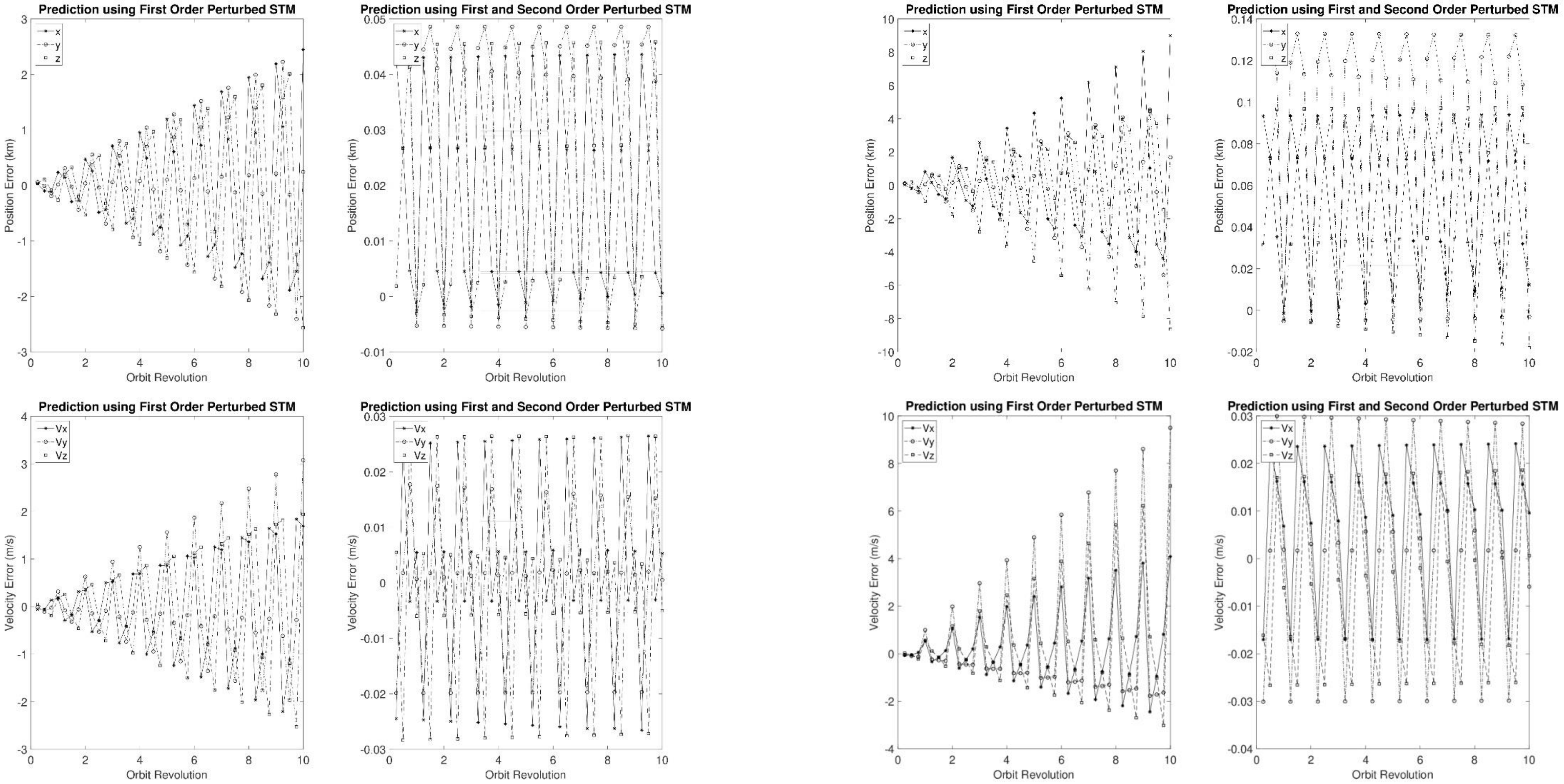


Fig: Linear prediction error improvement of states of J_2 perturbed LEO and MEO orbit using Second Order State Transition Tensor derived using Analytic Continuation technique

Results (continued)

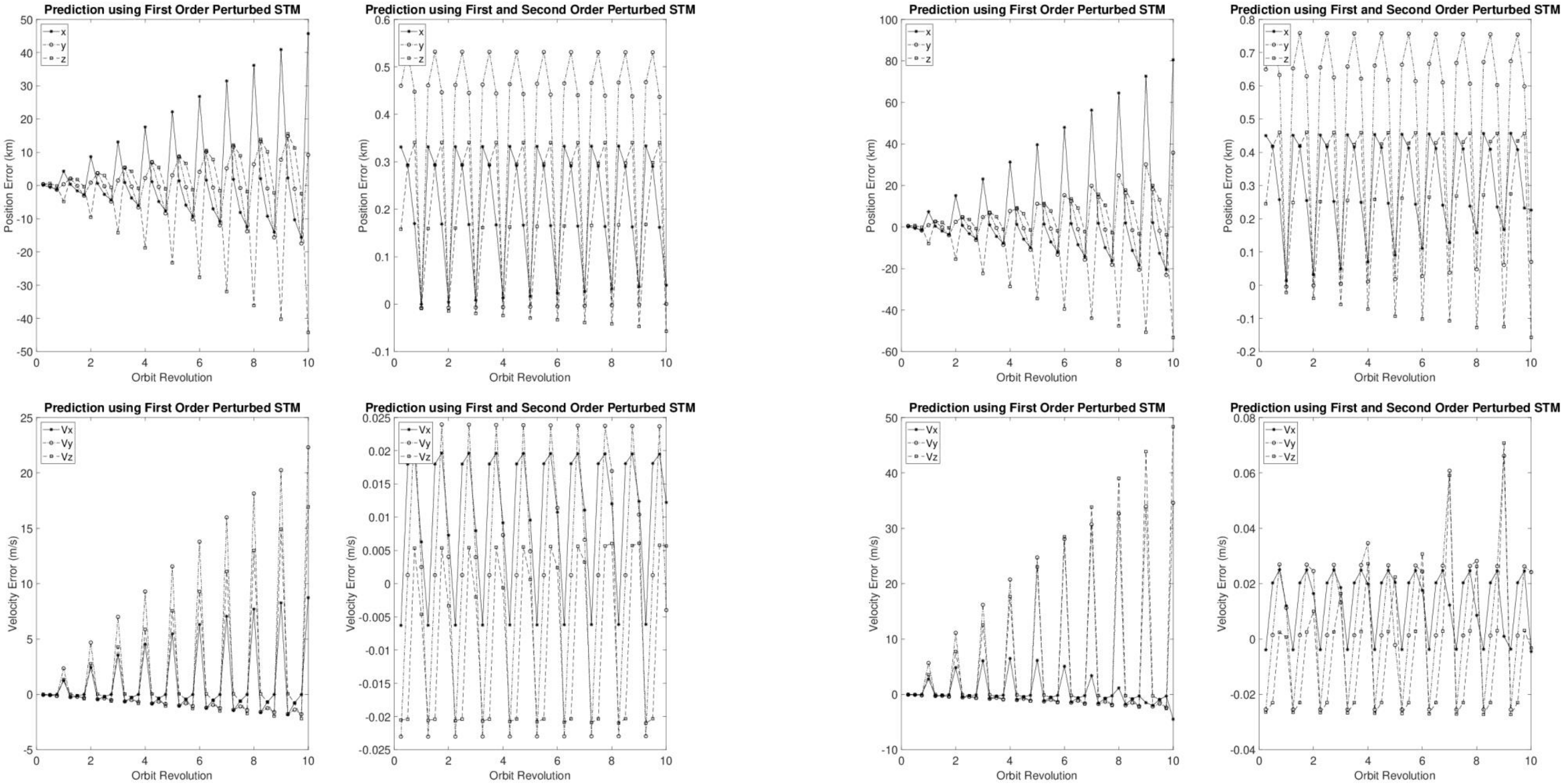


Fig: Linear prediction error improvement of states of J_2 perturbed GTO and HEO orbit using Second Order State Transition Tensor derived using Analytic Continuation technique

Publications

- Tasif, T.H., Elgohary, T.A.: A high order analytic continuation technique for the perturbed two-body problem state transition matrix, Advances in Astronautical Sciences: **AAS/AIAA Space Flight Mechanics Meeting** (2019)
- Tasif, T.H., Elgohary, T.A.: An adaptive analytic continuation technique for the computation of the higher order state transition tensors for the perturbed two-body problem, **AIAA Scitech 2020** Forum, p. 0958 (2020)
- Tasif, T.H., Elgohary, T.A.: An adaptive analytic continuation method for computing the perturbed two-body problem state transition matrix, **The Journal of the Astronautical Sciences** (2020), **In Press.**

Conclusions and Future Work

- Implementation of Spherical Harmonics Gravity model on State Transition Matrix and Higher Order State Transition Tensor is under development now.
- Atmospheric drag model will be implemented on Higher Order State Transition Tensors.
- The results of the current research work will be extended to solve uncertainty quantification of states over time and perturbed Multi Revolution Lambert's Problem.